

# THE CARLITZ SHTUKA

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**ABSTRACT.** Recently we have used the Carlitz exponential map to define a finitely generated submodule of the Carlitz module having the right properties to be a function field analogue of the group of units in a number field. Similarly, we constructed a finite module analogous to the class group of a number field.

In this short note more algebraic constructions of these “unit” and “class” modules are given and they are related to Ext modules in the category of shtukas.

## 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

**1.1. Notation.** Let  $k$  be a finite field of  $q$  elements. Without mention to the contrary schemes are understood to be over  $\mathrm{Spec} k$  and tensor products over  $k$ .

Let  $t$  be the standard coordinate on the projective line  $\mathbf{P}^1$  over  $k$ , let  $F = k(t)$  the function field of  $\mathbf{P}^1/k$  and let  $A = k[t]$  the ring of functions regular away from the “point at infinity”  $\infty \in \mathbf{P}^1$ .

Let  $X$  be a smooth projective geometrically connected curve over  $k$  and  $X \rightarrow \mathbf{P}^1$  a surjective map. Denote the function field of  $X$  by  $K$ . Let  $Y \subset X$  be the inverse image of  $\mathrm{Spec} A = \mathbf{P}^1 - \infty$ .

### 1.2. The Carlitz module.

**Definition 1.** The Carlitz module is the functor

$$C_0: \{\mathrm{Spec} A\text{-schemes}\} \rightarrow \{A\text{-modules}\}$$

which associates to a scheme  $S$  over  $\mathrm{Spec} A$  the  $A$ -module  $C(S)$  given by  $C(S) = \Gamma(S, \mathcal{O}_S)$  as a  $k$ -vector space, with  $A$ -module structure

$$\varphi: A \rightarrow \mathrm{End}_k \Gamma(S, \mathcal{O}_S): t \mapsto (c \mapsto tc + c^q).$$

The functor  $C_0$  is in many ways an analogue of the multiplicative group

$$\mathbf{G}_m: \{\mathrm{Spec} \mathbf{Z}\text{-schemes}\} \rightarrow \{\mathbf{Z}\text{-modules}\}: S \mapsto \Gamma(S, \mathcal{O}_S)^\times.$$

Yet, in contrast with Dirichlet’s unit theorem we have the following negative result:

**Proposition 1** (Poonen [8]). *The  $A$ -module  $C_0(Y)$  is not finitely generated.*  $\square$

**1.3. A construction using the Carlitz exponential.** In [9] we have used the Carlitz exponential map to cut out a canonical finitely generated sub- $A$ -module from  $C_0(Y)$ . We recall and reformulate this construction.

A simple recursion shows that there is a unique power series  $\exp x$  in  $F[[x]]$  which is of the form

$$\exp x = x + e_1 x^q + e_2 x^{q^2} + \dots$$

and which satisfies

$$(1) \quad \exp(tx) = t \exp x + (\exp x)^q.$$

This power series is called the *Carlitz exponential*. It is entire and for every point  $z$  of  $X \setminus Y$  it defines an  $A$ -linear map  $\exp: K_z \rightarrow C_0(K_z)$ .

Note that  $U \mapsto C_0(U)$  defines a (Zariski) sheaf of  $A$ -modules on  $Y$ . We extend it to a sheaf  $C$  on  $X$  as follows:

$$C(U) := \left\{ (c, (\gamma_z)_z) \in C_0(U \cap Y) \times \prod_{z \in U \setminus Y} K_z \mid \forall z \exp \gamma_z = c \right\}.$$

One verifies easily that this indeed defines a sheaf on  $X$ . The main result of [9] can be restated as follows:

**Proposition 2.**

- (1)  $H^0(X, C)$  is a finitely generated  $A$ -module;
- (2)  $H^1(X, C)$  is a finite  $A$ -module.

In §2 we will show how to deduce this result from [9].

In particular, the image of the restriction map  $C(X) \rightarrow C(Y) = C_0(Y)$  is a canonical finitely generated submodule of  $C_0(Y)$ , it is a Carlitz analogue of the group of units in a number field. Similarly,  $H^1(X, C)$  is a Carlitz analogue of the class group of a number field (this should of course be compared with the isomorphism  $H^1(\mathcal{O}_L, \mathbf{G}_m) = \text{Pic } \mathcal{O}_L$ ).

We *do* need to pass to the completed curve  $X$  to get something interesting: By Poonen's theorem  $H^0(Y, C_0)$  is not finitely generated, and since  $C_0 \cong \mathcal{O}_Y$  as sheaves of abelian groups we have that  $H^1(Y, C_0) = 0$ .

Unfortunately the above definition of the sheaf  $C$  is analytic in nature, and it would be desirable to have a purely algebraic description of  $C$ . The aim of this paper is to provide such a description, as well as a more "motivic" interpretation of it.

**1.4. An algebraic description of the sheaf  $C$ .** For an integer  $n$ , denote by  $\mathcal{O}_{\mathbf{P}^1}(n\infty)$  the sheaf of functions on  $\mathbf{P}^1$  that have a pole of order at most  $n$  at  $\infty$  and by  $\mathcal{O}_X(n\infty)$  its pullback over  $X \rightarrow \mathbf{P}^1$ .

**Theorem 1.** *There is a short exact sequence of sheaves of  $A$ -modules on  $X$*

$$(2) \quad 0 \longrightarrow \mathcal{O}_X \otimes A \xrightarrow{\partial} \mathcal{O}_X(\infty) \otimes A \longrightarrow C \longrightarrow 0$$

where

$$\partial: f \otimes a \mapsto f \otimes ta - (tf + f^q) \otimes a.$$

The proof of this theorem will be given in section §3.

**1.5. Interpretation in terms of shtukas.** The short exact sequence of Theorem 1 can be reinterpreted in terms of shtukas.

For any  $k$ -scheme  $S$  denote by  $S_A$  the base change of  $S$  to  $\text{Spec } A$  and by  $\tau_A: S_A \rightarrow S_A$  the base change of the  $q$ -th power Frobenius endomorphism  $\tau: S \rightarrow S$ .

**Definition 2.** A (right)  $A$ -shtuka on a  $k$ -scheme  $S$  is a diagram

$$\mathcal{M} = \left[ \mathcal{M} \xrightarrow{\sigma} \mathcal{M}' \xleftarrow{j} \tau_A^* \mathcal{M} \right]$$

of quasi-coherent  $\mathcal{O}_{S_A}$ -modules.

With the obvious notion of morphism, the shtukas on  $S$  form an  $A$ -linear abelian category. In particular, if  $\mathcal{M}_1$  and  $\mathcal{M}_2$  are two shtukas on  $S$  then the Yoneda extension groups  $\text{Ext}^i(\mathcal{M}_1, \mathcal{M}_2)$  are  $A$ -modules.

We have a natural isomorphism of sheaves of  $\mathcal{O}_X$ -modules

$$\tau^* \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X,$$

and will identify source and target in what follows.

If  $\mathcal{F}$  is a coherent sheaf of  $\mathcal{O}_X$ -modules and  $M$  an  $A$ -module we denote by  $\mathcal{F} \boxtimes M$  the coherent sheaf of  $\mathcal{O}_{X \times \text{Spec}(A)}$ -modules

$$\mathcal{F} \boxtimes M = \text{pr}_1^* \mathcal{F} \otimes_{\mathcal{O}_{X \times \text{Spec } A}} \text{pr}_2^* \tilde{M}$$

where  $\text{pr}_1$  and  $\text{pr}_2$  denote the projections from  $X \times \text{Spec } A$  to  $X$  and  $\text{Spec } A$  respectively.

**Definition 3.** The *unit shtuka* on  $X$  is defined to be the shtuka

$$1 = \left[ \mathcal{O}_X \boxtimes A \xrightarrow{1} \mathcal{O}_X \boxtimes A \xleftarrow{1} \tau^* \mathcal{O}_X \boxtimes A \right].$$

**Definition 4.** We define the *Carlitz shtuka* on  $X$  to be the shtuka

$$C = \left[ \mathcal{O}_X \boxtimes A \xrightarrow{\sigma} \mathcal{O}_X(\infty) \boxtimes A \xleftarrow{1} \tau^* \mathcal{O}_X \boxtimes A \right]$$

with

$$\sigma = 1 \otimes t - t \otimes 1.$$

The following is essentially a formal consequence of Theorem 1:

**Theorem 2.** *There are natural isomorphisms*

$$\mathrm{Ext}^i(\mathbf{1}, \mathcal{C}) \xrightarrow{\sim} H^{i-1}(X, \mathcal{C})$$

for all  $i$ .

The proof will be given in §4.

#### 1.6. Remarks.

**Remark 1.** Our notion of shtuka is the same as the one in V. Lafforgue [5]. It is similar to the one used by Drinfeld [3] and L. Lafforgue [4], but of a more arithmetic nature. Rather than compactifying the “coefficients”  $\mathrm{Spec} A$  to a complete curve, we compactify the “base”  $\mathbf{A}_k^1$  to  $\mathbf{P}_k^1$ .

**Remark 2.** Shtukas are function field toy models for (conjectural) mixed motives. The Carlitz shtuka  $\mathcal{C}$  is an analogue of the Tate motive  $\mathbf{Z}(1)$  and Theorem 2 should be compared with the isomorphisms

$$\mathrm{Ext}_X^1(\mathbf{1}, \mathbf{Z}(1)) = \Gamma(X, \mathcal{O}_X^\times) \quad \text{and} \quad \mathrm{Ext}_X^2(\mathbf{1}, \mathbf{Z}(1)) = \mathrm{Pic} X$$

from motivic cohomology, see for example [6, p. 25].

**Remark 3.** In the  $(\infty\text{-adic})$  “class number formula” proven in [10], the  $A$ -modules  $H^0(X, \mathcal{C})$  and  $H^1(X, \mathcal{C})$  play a role analogous to the groups of units and the class group in the classical class number formula. In the guise of  $\mathrm{Ext}^1(\mathbf{1}, \mathcal{C})$  and  $\mathrm{Ext}^2(\mathbf{1}, \mathcal{C})$  they play a similar role in V. Lafforgue’s result [5] on  $(v\text{-adic}, v \neq \infty)$  special values.

**Remark 4.** For any  $m$  there is a natural isomorphism  $\tau^* \mathcal{O}_X(m\infty) \xrightarrow{\sim} \mathcal{O}_X(qm\infty)$ . So in the definition of the Carlitz shtuka one could twist both line bundles with  $\mathcal{O}_X(-n\infty)$  for some  $n \geq 0$  to obtain

$$\mathcal{O}_X(-n\infty) \boxtimes A \xrightarrow{\sigma} \mathcal{O}_X((1-n)\infty) \boxtimes A \xleftarrow{1} \tau^* \mathcal{O}_X(-n\infty) \boxtimes A.$$

The same results with the same proofs hold for this shtuka. We have chosen  $n = 0$  in our definition somewhat arbitrarily, distinguishing it from the other choices only by its minimality.

**Remark 5.** We have treated in this note only a very special case. One should try to obtain similar results for higher rank Drinfeld modules over general Drinfeld rings  $A$ , and even for the abelian  $t$ -modules of Anderson [1]. Unfortunately it seems that these generalizations are not without difficulty, and even for higher rank Drinfeld modules it is not clear to me what the precise statement should be.

**1.7. Acknowledgements.** This work has been inspired by work of Anderson and Thakur [2], Woo [11], Papanikolas and Ramachandran [7], and V. Lafforgue [5]. The author is grateful to David Goss for his feedback and constant encouragement, and to the referee for several useful suggestions.

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## 2. THE COHOMOLOGY OF THE SHEAF $C$

In this section we show how the modules  $H^0(X, C)$  and  $H^1(X, C)$  compare with the modules studied in [9]. We recall the main result of *loc. cit.* Consider the map

$$\delta: C_0(Y) \times \prod_{z \in X \setminus Y} K_z \rightarrow \prod_{z \in X \setminus Y} C_0(K_z): (c, (\gamma_z)_z) \mapsto (c - \exp \gamma_z)_z.$$

**Theorem 3** ([9]).  *$\ker \delta$  is a finitely generated  $A$ -module and  $\operatorname{coker} \delta$  is a finite  $A$ -module.*  $\square$

We now show that  $H^0(X, C)$  and  $H^1(X, C)$  coincide with the modules  $\ker \delta$  (“ $\exp^{-1} C(R)$ ” in the notation of *loc. cit.*) and  $\operatorname{coker} \delta$  (“ $H_R$ ”) above, and hence that Proposition 2 follows from Theorem 3.

**Lemma 1.** *There is an exact sequence of  $A$ -modules*

$$0 \longrightarrow C(X) \longrightarrow C_0(Y) \times \prod_{z \in X \setminus Y} K_z \xrightarrow{\delta} \prod_{z \in X \setminus Y} C_0(K_z) \longrightarrow H^1(X, C) \longrightarrow 0.$$

*Proof.* Denote by  $i: Y \rightarrow X$  and by  $i_z: \{z\} \rightarrow X$  the inclusions of  $Y$  and the points  $z$  in  $X$ . Then the following sequence of sheaves on  $X$  is exact:

$$(3) \quad 0 \longrightarrow C \longrightarrow i_* C_0 \times \prod_{z \in X \setminus Y} i_{z,*} K_z \longrightarrow \prod_{z \in X \setminus Y} i_{z,*} C_0(K_z) \longrightarrow 0.$$

(Here the middle map is the difference of the natural map and the map induced by  $\exp$ .) Left exactness follows from the definition of  $C$ . For right exactness, one uses the fact that for all  $z \in X \setminus Y$  we have  $C_0(K_z) = C_0(K) + \exp K_z$  (which follows, for example, from Corollary 1 below).

Note that  $H^1(X, i_* C_0) = H^1(Y, C_0) = 0$  so that the desired exact sequence is precisely the long exact sequence of cohomology obtained from taking global sections in (3).  $\square$

## 3. PROOF OF THEOREM 1

3.1. **Away from  $\infty$ .** Let  $R$  be an  $A$ -algebra. Denote by  $\alpha$  the  $A$ -linear map

$$\alpha: R \otimes A \rightarrow C_0(R): r \otimes a \mapsto \varphi(a)(r).$$

**Proposition 3.** *The sequence of  $A$ -modules*

$$(4) \quad 0 \longrightarrow R \otimes A \xrightarrow{\partial} R \otimes A \xrightarrow{\alpha} C_0(R) \longrightarrow 0$$

*is exact.*

*Proof.* Straightforward.  $\square$

In particular this provides the desired short exact sequence of sheaves (2) on the affine curve  $Y \subset X$ . In the following paragraphs we will extend it to the whole of  $X$ .

3.2. **Inversion of the exponential map.** Let  $z \in X \setminus Y$  and let  $|\cdot|$  be an absolute value on  $K_z$  inducing the  $z$ -adic topology, so in particular  $|t| > 1$ .

**Lemma 2.** *For all  $x \in K_z$  with  $|x| < |t|^{q/(q-1)}$  we have  $|\exp x - x| < |x|$ .*

*Proof.* Write  $\exp x = \sum_{i=0}^{\infty} e_i x^{q^i}$ . It follows from (1) and from  $e_0 = 1$  that for all  $i$  we have  $|e_i| = |t|^{-iq^i}$ . From this one deduces that for all  $i > 0$  and all  $x$  with  $|x| < |t|^{q/(q-1)}$  the inequality  $|e_i x^{q^i}| < |x|$  holds. Hence  $|\exp x - x| = |\sum_{i>0} e_i x^{q^i}| < |x|$ , as claimed.  $\square$

**Corollary 1.** *For all  $m \leq 1$  the exponential map restricts to a  $k$ -linear isomorphism  $t^m \mathcal{O}_{X,z}^{\wedge} \rightarrow t^m \mathcal{O}_{X,z}^{\wedge}$ .*  $\square$

We denote its inverse by  $\log$ .

3.3. **Near  $\infty$ .** Let  $z \in X \setminus Y$ . Consider the  $A$ -linear map

$$\lambda: t\mathcal{O}_{X,z}^{\wedge} \otimes A \rightarrow K_z: f \otimes a \mapsto a \log f.$$

**Proposition 4.** *The sequence of  $A$ -modules*

$$(5) \quad 0 \longrightarrow \mathcal{O}_{X,z}^{\wedge} \otimes A \xrightarrow{\partial} t\mathcal{O}_{X,z}^{\wedge} \otimes A \xrightarrow{\lambda} K_z \longrightarrow 0$$

*is exact.*

*Proof.* Denote by  $\mu$  the multiplication map

$$\mu: t\mathcal{O}_{X,z}^{\wedge} \otimes A \rightarrow K_z: f \otimes a \mapsto af.$$

Using the identity (1) one verifies that the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathcal{O}_{X,z}^\wedge \otimes A & \xrightarrow{1 \otimes t - t \otimes 1} & t\mathcal{O}_{X,z}^\wedge \otimes A & \xrightarrow{\mu} & K_z \longrightarrow 0 \\
& & \downarrow \exp \otimes \text{id} & & \downarrow \exp \otimes \text{id} & & \downarrow \text{id} \\
0 & \longrightarrow & \mathcal{O}_{X,z}^\wedge \otimes A & \xrightarrow{\partial} & t\mathcal{O}_{X,z}^\wedge \otimes A & \xrightarrow{\lambda} & K_z \longrightarrow 0
\end{array}$$

commutes. The vertical arrows are isomorphisms by Corollary 1 and since the top sequence is exact the same holds for the bottom sequence.  $\square$

**3.4. Conclusion.** It is now a purely formal matter to deduce Theorem 1 from Propositions 3 and 4:

*Proof of Theorem 1.* Clearly the map

$$\mathcal{O}_X \otimes A \xrightarrow{\partial} \mathcal{O}_X(\infty) \otimes A$$

is injective. We need to construct an isomorphism  $\text{coker } \partial \xrightarrow{\sim} C$ .

For every open  $U \subset X$  and every integer  $m$  we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(m\infty)(U) \longrightarrow \mathcal{O}_X(U \cap Y) \times t^m \prod_z \mathcal{O}_{X,z}^\wedge \xrightarrow{\delta} \prod_z K_z$$

where the products range over  $z \in U \setminus Y$  and where  $\delta(f, g) := f - g$ . If moreover  $U$  is affine then  $\delta$  is surjective and we obtain a short exact sequence which we denote by  $E(m)$ .

Now, for an affine  $U$ , consider the map of exact sequences

$$\partial: E(0) \otimes A \rightarrow E(1) \otimes A.$$

It is injective in all three positions. Using (4) and (5) one sees that the quotient is isomorphic with a short exact sequence

$$0 \longrightarrow (\text{coker } \partial)(U) \longrightarrow C_0(U \cap Y) \times \prod_z K_z \longrightarrow \prod_z C_0(K_z) \longrightarrow 0,$$

the last map being  $(c, f) \mapsto c - \exp f$ . This provides an isomorphism  $(\text{coker } \partial)(U) \xrightarrow{\sim} C(U)$  for every affine open  $U \subset X$ , and clearly these glue to an isomorphism of sheaves. This proves Theorem 1.  $\square$

#### 4. PROOF OF THEOREM 2

Let  $S$  be a  $k$ -scheme. For any  $\mathcal{O}_{S_A}$ -module  $\mathcal{F}$  we denote by  $\tau$  the canonical isomorphism of  $S_A$ -sheaves

$$\tau: \mathcal{F} \longrightarrow \tau_A^* \mathcal{F}$$

which is  $A$ -linear but generally *not*  $\mathcal{O}_{S_A}$ -linear. If

$$\mathcal{M} = \left[ \mathcal{M} \xrightarrow{\sigma} \mathcal{M}' \xleftarrow{j} \tau^* \mathcal{M} \right]$$

is a shtuka on  $S$  then we denote by  $\mathcal{M}^\bullet$  the complex of  $S_A$ -sheaves

$$\mathcal{M} \xrightarrow{\partial} \mathcal{M}'$$

in degrees 0 and 1, with  $\partial = \sigma - j \circ \tau$ .

The following Proposition can be found implicitly in [5].

**Proposition 5.** *For all  $i$  and all  $\mathcal{M}$  there are natural isomorphisms*

$$\mathrm{Ext}^i(\mathbf{1}, \mathcal{M}) = \mathbb{H}^i(S_A, \mathcal{M}^\bullet),$$

*functorial in  $\mathcal{M}$  and in  $S$ .*

Before giving a proof, we first deduce Theorem 2 from this proposition.

*Proof of Theorem 2.* Applying the Proposition to  $S = X$  and  $\mathcal{M} = \mathcal{C}$  we find

$$\mathrm{Ext}^i(\mathbf{1}, \mathcal{C}) = \mathbb{H}^i(X_A, \mathcal{O}_X \boxtimes A \xrightarrow{\partial} \mathcal{O}_X(\infty) \boxtimes A).$$

The latter is isomorphic with

$$\mathbb{H}^i(X, \mathcal{O}_X \otimes A \xrightarrow{\partial} \mathcal{O}_X(\infty) \otimes A)$$

which by Theorem 1 is isomorphic with  $\mathbb{H}^{i-1}(X, \mathcal{C})$ .  $\square$

*Proof of Proposition 5.* We will first establish a canonical isomorphism for  $i = 0$ , and then conclude the general case by a purely formal argument.

A homomorphism  $\mathbf{1} \rightarrow \mathcal{M}$  of shtukas on  $S$  is a commutative diagram

$$\begin{array}{ccccc} \mathcal{O}_{S_A} & \xrightarrow{1} & \mathcal{O}_{S_A} & \xleftarrow{1} & \tau^* \mathcal{O}_{S_A} \\ \downarrow f & & \downarrow f' & & \downarrow \tau^* f \\ \mathcal{M} & \xrightarrow{\sigma} & \mathcal{M}' & \xleftarrow{j} & \tau^* \mathcal{M} \end{array}$$

Clearly the homomorphism is uniquely determined by  $f \in \Gamma(S_A, \mathcal{M})$ , and an  $f \in \Gamma(S_A, \mathcal{M})$  extends to a homomorphism of shtukas if and only if  $\partial f = 0$ . So we obtain an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathbf{1}, \mathcal{M}) \longrightarrow \Gamma(S_A, \mathcal{M}) \xrightarrow{\partial} \Gamma(S_A, \mathcal{M}')$$

and hence an isomorphism

$$\mathrm{Hom}(\mathbf{1}, \mathcal{M}) = \mathbb{H}^0(S_A, \mathcal{M}^\bullet).$$

Now any shtuka

$$\mathcal{I} = \left[ \mathcal{I} \xrightarrow{\sigma} \mathcal{I}' \xleftarrow{j} \tau^* \mathcal{I} \right]$$

with  $\mathcal{I}$  and  $\mathcal{I}'$  injective  $\mathcal{O}_{S_A}$ -modules is an injective object in the category of shtukas on  $S$ . So we can find an injective resolution  $\mathcal{I}^\bullet$  of the shtuka



$\mathcal{M}$  such that the resulting double complex  $\mathcal{I}^{\bullet\bullet}$  is an injective resolution of the complex  $\mathcal{M}^\bullet$ . We obtain a canonical isomorphism

$$\mathrm{Ext}^i(\mathbf{1}, \mathcal{M}) = \mathrm{H}^i(S_A, \mathcal{M}^\bullet)$$

for all  $i$ . □

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